Note on the Upper Tangency Condition

We first derive a test of the upper tangency condition for the mimicking-portfolio case. We then specialize the resulting expression to the case of traded factors only. Suppose that the K-vector f_t consists of some traded and some nontraded factors. Let R_t be an N-vector of excess returns (return spreads) on the basis assets. For a traded factor, the mimicking portfolio is, of course, simply the factor itself.

We define $Y_t = [f_t', \ R_t']'$ and its population mean and covariance matrix as

$$\mu = E[Y_t] \equiv \begin{bmatrix} \mu_f \\ \mu_R \end{bmatrix}, \tag{1}$$

$$V = \text{Var}[Y_t] \equiv \begin{bmatrix} V_f & V_{fR} \\ V_{Rf} & V_R \end{bmatrix}. \tag{2}$$

In the following analysis, we assume that V_f and V_R are invertible and that V_{Rf} is of full column rank. Consider the projection of f_t on R_t and a constant and denote the resulting mimicking-portfolio returns by $f_t^* = V_{fR}V_R^{-1}R_t \equiv AR_t$ with $\mu^* = E[f_t^*] = A\mu_R$ and $V^* = \text{Var}[f_t^*] = AV_RA' = V_{fR}V_R^{-1}V_{Rf}$. The initial costs of the mimicking portfolios are Ae, where $e_i = 1$ for basis asset returns in excess of the risk-free rate and $e_i = 0$ for differences in returns on two basis assets.*

A model's tangency portfolio is on the upper half of the minimum-variance frontier when

$$b = e'A'(AV_RA')^{-1}A\mu_R \geq 0.$$

Denote with hats the sample equivalents of the quantities above. Then, the upper tangency condition can be tested by considering $\hat{b} = e'\hat{A}'(\hat{A}\hat{V}_R\hat{A}')^{-1}\hat{A}\hat{\mu}_R$ and its associated t-statistic. We now derive the asymptotic distribution of $\hat{b}$.

*Alternatively, one could set the e vector equal to a vector of ones. This requires assuming that return spreads, such as HML, can be viewed as long $1 in the risk-free asset and $1 on each side of the spread. Then, HML is the excess return on that unit investment factor.

Note that

$$\begin{aligned} b &= e'A'(AV_RA')^{-1}A\mu_R \\ &= e'A'V^{*-1}\mu^\star \\ &= e'V_R^{-1}\beta^*\mu^\star, \end{aligned} \tag{3}$$

where $\beta^* = V_{Rf}V^{*-1}$.

It is easy to show that

$$\frac{\partial b}{\partial \mu_f'} = 0_K', \tag{4}$$

$$\frac{\partial b}{\partial \mu_R'} = e'A'V^{*-1}A. \tag{5}$$

Moreover,

$$\frac{\partial b}{\partial \text{vec}(V)'} = (\mu^{*\prime} \otimes e'V_R^{-1})\frac{\partial \text{vec}(\beta^*)}{\partial \text{vec}(V)'} + (\mu^{*\prime}\beta^{*\prime} \otimes e')\frac{\partial \text{vec}(V_R^{-1})}{\partial \text{vec}(V)'} + e'V_R^{-1}\beta^*\frac{\partial \mu^*}{\partial \text{vec}(V)'}. \tag{6}$$

After some algebra, we obtain

$$\frac{\partial \text{vec}(\beta^*)}{\partial \text{vec}(V)'} = [0_{K\times K},\ V^{*-1}A] \otimes [-\beta^*,\ \beta^*A] + [V^{*-1},\ 0_{K\times N}] \otimes [0_{N\times K},\ I_N - \beta^*A], \tag{7}$$

$$\frac{\partial \text{vec}(V_R^{-1})}{\partial \text{vec}(V)'} = [0_{N\times K},\ V_R^{-1}] \otimes [0_{N\times K},\ -V_R^{-1}], \tag{8}$$

$$\frac{\partial (\mu^*)}{\partial \text{vec}(V)'} = [I_K,\ 0_{K\times N}] \otimes [0_K',\ \mu_R'V_R^{-1}] - [0_K',\ \mu_R'V_R^{-1}] \otimes [0_{K\times K},\ A]. \tag{9}$$

Putting everything together, Equation (6) becomes

$$\begin{aligned} \frac{\partial b}{\partial \text{vec}(V)'} = {} & [0_K',\ \mu^{*\prime}V^{*-1}A] \otimes [-e'A'V^{*-1},\ e'A'V^{*-1}A] \\ & + [\mu^{*\prime}V^{*-1},\ -\mu^{*\prime}V^{*-1}A] \otimes [0_K',\ e'V_R^{-1}] \\ & - [\mu^{*\prime}V^{*-1},\ \mu_R'V_R^{-1}] \otimes [0_K',\ e'A'V^{*-1}A] \\ & + [e'A'V^{*-1},\ 0_N'] \otimes [0_K',\ \mu_R'V_R^{-1}]. \end{aligned} \tag{10}$$

The proof relies on the fact that $\hat{b}$ is a smooth function of $\hat{\mu}$ and $\hat{V}$. Therefore, once we have the asymptotic distribution of $\hat{\mu}$ and $\hat{V}$, we can use the delta method to obtain the asymptotic distribution of $\hat{b}$. Let

$$\varphi = \begin{bmatrix} \mu \\ \mathrm{vec}(V) \end{bmatrix}, \qquad \hat{\varphi} = \begin{bmatrix} \hat{\mu} \\ \mathrm{vec}(\hat{V}) \end{bmatrix}. \tag{11}$$

We first note that $\hat{\mu}$ and $\hat{V}$ can be written as the GMM estimator that uses the moment conditions $E[r_t(\varphi)] = 0_{(N+K)(N+K+1)}$, where

$$r_t(\varphi) = \begin{bmatrix} Y_t - \mu \\ \mathrm{vec}((Y_t - \mu)(Y_t - \mu)' - V) \end{bmatrix}. \tag{12}$$

Since this is an exactly identified system of moment conditions, it is straightforward to verify that under the assumption that Y_t is stationary and ergodic with finite fourth moment, we have

$$\sqrt{T}(\hat{\varphi} - \varphi) \overset{A}{\sim} N(0_{(N+K)(N+K+1)}, S_0), \tag{13}$$

where

$$S_0 = \sum_{j=-\infty}^{\infty} E[r_t(\varphi) r_{t+j}(\varphi)']. \tag{14}$$

Note that S_0 is a singular matrix as $\hat{V}$ is symmetric, so there are redundant elements in $\hat{\varphi}$. We could have written $\hat{\varphi}$ as $[\hat{\mu}', \ \mathrm{vech}(\hat{V})']'$, but the results are the same under both specifications.

Using the delta method, the asymptotic distribution of $\hat{b}$ is given by

$$\begin{aligned} \sqrt{T}(\hat{b} - b) & \overset{A}{\sim} N\left(0, \left[\frac{\partial b}{\partial \varphi'}\right] S_0 \left[\frac{\partial b}{\partial \varphi'}\right]'\right) \\ & \overset{A}{\sim} N\left(0, E[g_t^2]\right), \end{aligned} \tag{15}$$

where

$$g_t = e'A'V^{*-1}(f_t^* - \mu^*)[1 - y_t - u_t] + e'A'V^{*-1}(v_t - u_t)\eta_t + e'V_R^{-1}(R_t - \mu_R)y_t + b, \tag{16}$$

where $v_t = \mu_R' V_R^{-1}(R_t - \mu_R)$, $u_t = \mu^{*\prime} V^{*-1}(f_t^* - \mu^*)$, $y_t = \mu^{*\prime} V^{*-1}\eta_t$, and $\eta_t = (f_t - \mu_f) - A(R_t - \mu_R)$.

When all the factors are traded, the expression above simplifies by noting that $f_t^* = f_t$, $\mu^* = \mu_f$, $V^* = V_f$, $\eta_t = 0_K$, $y_t = 0$, and $Ae = q$, where q are the initial costs of the K factors. We then have

$$b = q' V_f^{-1} \mu_f$$

and

$$g_t = q' V_f^{-1} (f_t - \mu_f)[1 - \mu_f' V_f^{-1} (f_t - \mu_f)] + b. \tag{17}$$