Journal of Financial Econometrics, 2020, Vol. 18, No. 4, 729–735
doi: 10.1093/jjfinec/nbaa001
Advance Access Publication Date: 20 February 2020
Commentary

Comment on: Pseudo-True SDFs in Conditional Asset Pricing Models

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accepted January 21, 2020

Asset pricing models are, at best, approximations of reality and are bound to be misspecified. However, it can still be useful to empirically evaluate the degree of model misspecification and the relative performance of competing asset pricing models using actual data. In their seminal paper, Hansen and Jagannathan (1997, HJ hereafter) propose two measures of model misspecification, which are now routinely used for parameter estimation, specification testing, and model selection. The first one measures the distance between the proposed stochastic discount factor (SDF) and the set of admissible SDFs (i.e., the set of SDFs that price a given set of test assets correctly). The second one measures the distance between the proposed SDF and the set of nonnegative admissible SDFs. Since the first measure does not impose the nonnegativity constraint (no-arbitrage condition) on the set of admissible SDFs, whereas the second one does, we refer the first measure as the unconstrained HJ-distance and the second one as the constrained HJ-distance.

The theoretical and large-sample statistical properties of the unconstrained and constrained HJ-distances in an *unconditional* setting have been explored in depth. Under correctly specified and linear SDFs, Jagannathan and Wang (1996) characterize the large-sample behavior of the sample squared unconstrained HJ-distance.[1] For general SDFs, Hansen, Heaton, and Luttmer (1995) derive the limiting distribution of the sample unconstrained HJ-distance under model misspecification. Gospodinov, Kan, and Robotti (2016) provide an in-depth analysis of the population constrained HJ-distance for the case of linear SDFs under a multivariate elliptical assumption on the factors and the returns. In addition, for general SDFs, they characterize the limiting behavior of the sample constrained HJ-distance under correctly specified and misspecified models.[2] They show the equivalence of the asymptotic distributions of the sample constrained and unconstrained HJ-distance tests and point out that the specification test developed for the sample unconstrained

1 Parker and Julliard (2005) extend the results in Jagannathan and Wang (1996) to the case of nonlinear models.

2 See also Hansen, Heaton, and Luttmer (1995) and Li, Xu, and Zhang (2010).

HJ-distance is also applicable to the sample constrained HJ-distance. In essence, under the null of a correctly specified model, the constraints are not binding and the two tests are asymptotically equivalent.[3] Furthermore, the limiting behaviors of the SDF parameter estimates and associated sample Lagrange multipliers in the unconstrained and constrained HJ-distance settings have been derived (see, e.g., Hansen, Heaton, and Luttmer, 1995; Hansen and Jagannathan, 1997; Kan and Robotti, 2008, 2009; Li, Xu, and Zhang, 2010; Gospodinov, Kan, and Robotti, 2013, 2016.) Gospodinov, Kan, and Robotti (2014, 2019) study the limiting distributions of the sample unconstrained HJ-distance and the associated SDF parameter and Lagrange multiplier estimates in the presence of potential model misspecification and identification failure that is caused by the presence of spurious ("useless") factors in the analysis. Formal asymptotic tests of pairwise and multiple model comparison based on the unconstrained and constrained HJ-distances have also been developed in the literature. (See Kan and Robotti, 2009; Li, Xu, and Zhang, 2010; Gospodinov, Kan, and Robotti, 2013.) Finally, the exact distributions of the sample HJ-distances under correctly specified and misspecified models have been little explored. The only exceptions are Kan and Zhou (2004) and Gospodinov, Kan, and Robotti (2016).

While, as emphasized above, much is known about performance evaluation and model selection based on the unconditional HJ-distance measure, a rigorous treatment of the HJ-distance metric in the presence of conditional moment restrictions is still in its infancy. This is unfortunate since asset pricing theory typically implies a set of conditional moment restrictions. For example, Dominguez and Lobato (2004) argue that, despite its computational attractiveness, the standard generalized method of moments (GMMs) approach of Hansen (1982) based on unconditional moment restrictions may result in efficiency losses and inconsistencies that arise from possible nonidentifiability of the parameters of interest by the unconditional moment restrictions even when the conditional moment restrictions identify the parameters. This article aims at shedding some light on these issues by analyzing the population and sampling properties of the so-called *conditional* HJ-distance.

Let R_{t+1} be the gross returns on N test assets at the end of time $t+1$, and let $\mathcal{I}_t$ be the information available to the investors at time t. Absence of arbitrage is equivalent to the existence of a scalar stochastic process $\{m_{t,t+1}\}$ such that the SDF $m_{t,t+1}$ between time t and time $t+1$ is positive, is in the linear space of random variables with finite second moment and measurable with respect to information $\mathcal{I}_{t+1}$, and satisfies the set of conditional moment restrictions

$$E[m_{t,t+1}R_{t+1}|\mathcal{I}_t] = 1_N, \tag{1}$$

where 1_N is an N-vector of ones. When the admissible SDF, $m_{t,t+1}$, is replaced with a candidate SDF, y_{t+1}, which depends on a p-vector of unknown parameters θ, we have two possibilities. When the candidate SDF is correctly specified and well identified, Equation (1) implies the set of N conditional moment restrictions

$$E[y_{t+1}(\theta_0)R_{t+1} - 1_N|\mathcal{I}_t] = 0_N, \tag{2}$$

where 0_N is an N-vector of zeros and θ_0 is the (unique) unknown true value of the SDF parameter vector. Consequently, we can think of $y_{t+1}(\theta_0)$ as the true SDF. In contrast, when

3 A similar result holds for the limiting distributions of the parameter estimates under correctly specified models which coincide with those for the unconstrained HJ-distance.

there does not exist a θ such that Equation (2) holds, then the candidate SDF is misspecified. In this scenario, θ_0 will not be unique any longer and is referred to as the *pseudo-true* value. This pseudo-true value of the parameter vector and the corresponding pseudo-true SDF will not only depend on the particular choice of asset pricing model, but also on the loss function whose first-order conditions are set to zero in optimization. The focus of the authors is on a specific loss function, the unconstrained HJ-distance measure in a conditional setting.

Let $e(\mathcal{I}_t,\theta)=E[y_{t+1}(\theta)R_{t+1}-1_N|\mathcal{I}_t]$ and $\Omega(\mathcal{I}_t)=E[R_{t+1}R'_{t+1}|\mathcal{I}_t]$, then the state-dependent conditional squared HJ-distance is given by

$$\delta^2(\mathcal{I}_t)=\min_{\theta} e(\mathcal{I}_t,\theta)'\Omega^{-1}(\mathcal{I}_t)e(\mathcal{I}_t,\theta). \quad (3)$$

The resulting (state-dependent) pseudo-true value, θ_t, $t=1,\dots,T$, could then be obtained as the solution to the first-order conditions

$$E\left[\frac{\partial y_{t+1}(\theta_t)}{\partial\theta}R'_{t+1}|\mathcal{I}_t\right]\Omega^{-1}(\mathcal{I}_t)e(\mathcal{I}_t,\theta_t)=0_p, \quad (4)$$

where 0_p is a p-vector of zeros, and the associated pseudo-true SDF is given by $y_{t+1}(\theta_t)$. Note that the metric in Equation (3) is truly conditional, in the sense that the minimized HJ-distance and associated pseudo-true values are state dependent. This is the definition of conditional HJ-distance proposed by Balduzzi and Robotti (2010) and subsequently employed by Fang, Ren, and Yuan (2011). A similar use of a state-dependent pseudo-true value has also been promoted by Gagliardini, Gourieroux, and Renault (2011). This said, this is not the measure the authors focus on in most of their population and econometric analyses. Instead, as in Gagliardini and Ronchetti (2020), the authors consider an average conditional squared HJ-distance of the form

$$\delta^2=\min_{\theta} E[e(\mathcal{I}_t,\theta)'\Omega^{-1}(\mathcal{I}_t)e(\mathcal{I}_t,\theta)]. \quad (5)$$

The pseudo-true values, θ^*, are then the solution to the first-order conditions

$$E\left\{E\left[\frac{\partial y_{t+1}(\theta^*)}{\partial\theta}R'_{t+1}|\mathcal{I}_t\right]\Omega^{-1}(\mathcal{I}_t)e(\mathcal{I}_t,\theta^*)\right\}=0_p, \quad (6)$$

and the associated pseudo-true SDF is given by $y_{t+1}(\theta^*)$. This is also the framework adopted in Gospodinov and Otsu (2012) and Proulx (2018). The average conditional HJ-distance in Equation (6) involves an unconditional expectation of the argument. Therefore, the proposed measure is constant. It is conditional only in the sense that the pricing errors are state dependent. This raises some important questions. What is the economic interpretation of this average conditional HJ-distance and of the corresponding pseudo-true SDF $y_{t+1}(\theta^*)$? In a sense, we lose the nice economic interpretation of HJ-distance provided by Hansen and Jagannathan (1997) in an unconditional setting. First, the resulting adjusted SDF $y_{t+1}(\theta^*)-\lambda(\theta^*,\mathcal{I}_t)'R_{t+1}$, where $\lambda(\theta^*,\mathcal{I}_t)=\Omega^{-1}(\mathcal{I}_t)e(\mathcal{I}_t,\theta^*)$ is the vector of time-varying Lagrange multipliers will not be the correct SDF conditionally. Second, this adjusted SDF will also not correctly price the assets unconditionally. In contrast, the adjusted SDF based on $y_{t+1}(\theta_t)$ will correctly price the assets conditionally and unconditionally. Furthermore, the maximum pricing error interpretation of the HJ-distance in an unconditional setting will break down when using $y_{t+1}(\theta^*)$ as the pseudo-true SDF. Overall, when the model is misspecified, one can always choose some objective function to define the pseudo-true parameters, but the original interpretation of HJ-distance may be compromised. It should also be emphasized that ranking the performance of competing asset pricing models based on δ^2 instead of $\delta^2(\mathcal{I}_t)$ could be problematic. For example, the potentially interesting task of determining how different models perform relative to each other over time would be obfuscated by the use of the unconditional average operator. The authors are clearly aware of these interpretation issues. However, it is not clear at this point how one can deal with a truly conditional HJ-distance measure such as $\delta^2(\mathcal{I}_t)$. Certainly, the econometrics of $\delta^2(\mathcal{I}_t)$ would be challenging.

The computation of the sample average conditional HJ-distance requires the nonparametric estimation of the conditional pricing error vector and the conditional second moments matrix of the gross returns on the test assets. When the asset pricing model is globally misspecified, the asymptotic covariance matrix of the estimator $\hat{\theta}$ of θ^* will need to account for misspecification uncertainty in addition to sampling uncertainty.[4] Since θ^* depends on the choice of weighting matrix, the asymptotic covariance of $\hat{\theta}$ will be sensitive to the chosen weighting scheme. In addition, under model misspecification, the asymptotic distribution of the underlying estimator will implicitly depend on the choice of states. Therefore, one may not want to be overly parsimonious in the number of states to include in the analysis. An immediate consequence of this is that when it comes to comparing alternative pricing models, their relative rankings may be sensitive to the state variables included in the analysis.

This article proposes to estimate θ^* using the smooth minimum distance (SMD) approach of Lavergne and Patilea (2013). The SMD approach avoids the need of trimming strategies that are typically required by classical local GMM estimators and, by allowing for fixed bandwidth asymptotics, is less sensitive to the curse of dimensionality related to the number of states.[5] The authors show that the SMD approach can be seen as a conditional extension of the jackknife GMM of Newey and Windmeijer (2009). This conditional extension of the jackknife GMM with kernel smoothing allows the authors to revisit the theory of GMM under misspecification (see Hall and Inoue, 2003) and to derive the asymptotic covariance matrix of $\hat{\theta}$ under milder assumptions than in Hall and Inoue (2003). The sieves minimum distance approach of Ai and Chen (2007) could also be employed to derive the limiting distribution of $\hat{\theta}$.[6] Similarly, Gagliardini and Ronchetti (2020, appendix C)

4 Gallant and White (1988) were the first to study GMM under globally misspecified models, but, as Hall and Inoue (2003) note, they did not treat the important case of a stochastic weighting matrix. However, Theorem 6.10 of White (1994) could be used to obtain asymptotic results under misspecified GMM with a stochastic weighting matrix.

5 In contrast, the optimal instruments (sieves-based) approach of Nagel and Singleton (2011), which is based on Hansen (1985), is highly sensitive to this curse of dimensionality. The local GMM approach of Gagliardini and Ronchetti (2020) also prevents the number of states from being too large. In addition, the consistency and asymptotic normality of their estimator critically hinge on the bandwidth parameter converging to zero as $T \to \infty$.

6 Although the results in Ai and Chen (2007) are based on an identity weighting matrix, their approach is robust in considering a stochastic weighting matrix such as the second moment matrix in the HJ-distance problem.

derive the limiting distribution of $\hat{\theta}$ under correctly specified as well as misspecified models using local GMM. The derivations for the correctly specified case are based on Gospodinov and Otsu (2012), while the ones for the misspecified case extend the unconditional setting of Hall and Inoue (2003) to a scenario in which the parameters are identified by a set of conditional moment restrictions (see also Anatolyev and Gospodinov, 2011).

As for most nonparametric analyses, several choices need to be made to render the theory operational. For example, the choice of kernel and bandwidths is likely to play an important role. This article actually considers two bandwidth parameters, one for the conditional moment restrictions and the other for the weighting matrix. The authors provide us with some preliminary ideas of how the proposed estimator fares in small samples. The results are encouraging and sometimes SMD seems to have better finite-sample properties than local GMM. However, more simulation experiments are needed to assess the reliability of the asymptotic approximations proposed in this article and to claim that the SMD approach works better than local GMM in small samples. In addition to the test asset returns and factors, we also have instruments that may contribute to an increase in sampling uncertainty relative, say, to the unconditional HJ-distance case. In particular, it would be insightful to also explore the size and power properties of the *t*-test associated with the SMD estimator under correctly specified and misspecified models.[7]

This article takes an important step toward understanding population and sampling issues in conditional moment restriction models. From an economic perspective, it would be desirable to shed further light on the role played by the pseudo-true SDF $y_{t+1}(\theta^*)$ in pricing, hedging, and forecasting problems. From a statistical perspective, more work needs to be done to assess the large- and small-sample properties of alternative nonparametric estimators in absolute and relative testing problems. For example, using a local GMM approach, Gagliardini and Ronchetti (2020) go beyond parameter estimates and associated asymptotic standard errors. They actually provide us with the limiting distributions of the sample average conditional HJ-distance under correctly specified and misspecified models. Consistent with Gospodinov and Otsu (2012), they show that an appropriately recentered sample (average) squared conditional HJ-distance is normally distributed even under the null of correct model specification. This is in sharp contrast with the unconditional case where it is well known that the sample unconstrained and constrained HJ-distances have weighted chi-squared limiting distributions.[8] Gagliardini and Ronchetti (2020) also provide pairwise model comparison tests based on the (average) conditional squared HJ-distance metric, effectively extending the results of Kan and Robotti (2009) and Gospodinov, Kan, and Robotti (2013) to a conditional setting. Furthermore, it would be valuable to accommodate multiple model comparison in the analysis. Finally, an important task will be to robustify the various tests statistics not only against global model misspecification but also against potential identification weakness caused by the presence of spurious factors and/or

7 For linear and well-identified SDFs in an unconditional HJ-distance setting, Kan and Robotti (2009) find that the misspecification robust *t*-test associated with the SDF parameter estimates is often conservative.

8 A weighted Chi-squared limiting distribution under the null also emerges in the conditional HJ-distance case that was studied by Fang, Ren, and Yuan (2011).

instruments in the analysis. To this end, the work of Antoine and Lavergne (2014) is promising.[9]

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