On Moments of Folded and Truncated Multivariate Normal Distributions

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Abstract

Recurrence relations for integrals that involve the density of multivariate normal distributions are developed. These recursions allow fast computation of the moments of folded and truncated multivariate normal distributions. Besides being numerically efficient, the proposed recursions also allow us to obtain explicit expressions of low order moments of folded and truncated multivariate normal distributions.

Keywords: Multivariate normal distribution; Folded normal distribution; Truncated normal distribution.
1 Introduction

Suppose \( X = (X_1, \ldots, X_n)^T \) follows a multivariate normal distribution with mean \( \mu \) and positive definite covariance matrix \( \Sigma \). We are interested in computing \( E(|X_1|^{k_1} \cdots |X_n|^{k_n}) \) and \( E(X_1^{k_1} \cdots X_n^{k_n} \mid a_i < X_i < b_i, \ i = 1, \ldots, n) \) for nonnegative integer values \( k_i = 0, 1, 2, \ldots \). The first expression is the moment of a folded multivariate normal distribution \( |X| = (|X_1|, \ldots, |X_n|)^T \). The second expression is the moment of a truncated multivariate normal distribution, with \( X_i \) truncated at the lower limit \( a_i \) and upper limit \( b_i \). In the second expression, some of the \( a_i \)'s can be \( -\infty \) and some of the \( b_i \)'s can be \( \infty \). When all the \( b_i \)'s are \( \infty \), the distribution is called the lower truncated multivariate normal, and when all the \( a_i \)'s are \( -\infty \), the distribution is called the upper truncated multivariate normal.

The folded univariate normal distribution was first introduced by Leone et al. (1961), and Elandt (1961) provides expressions for its moments. Psarakis and Panaretos (2001) generalize the folded distribution to the bivariate normal case and provide the moment generating function when \( \mu = 0 \). Recently, Chakraborty and Chatterjee (2013) introduce the folded multivariate normal distribution. They present the joint density, the moment generating function, and the mean and covariance matrix of \( |X| \). Unfortunately, as pointed out by Murthy (2015), the moment generating function as well as the mean and covariance matrix expressions given in Chakraborty and Chatterjee (2013) are incorrect. The moments of the folded multivariate normal distribution are simply the absolute moments of the multivariate normal distribution. When \( \mu = 0 \), there is a literature that provides explicit formulae for these absolute moments. Nabeya (1951) derives an explicit expression of the absolute moments for the bivariate normal case. Nabeya (1952) presents explicit expressions of the absolute moments for the trivariate normal case (up to 12th order, see also related results in Kamat (1953)). For the 4-variate case, Nabeya (1961) provides explicit expressions of some low order absolute moments. However, the computation of higher order absolute moments has been a challenge for \( n > 2 \) even when \( \mu = 0 \). When \( \mu \neq 0 \), we are unaware of any result that enables us to compute arbitrary order absolute moments of a multivariate normal distribution (except when \( n = 1 \)).

There is a long and rich literature on truncated normal distributions. For \( n = 1 \), Cohen (1991) provides a comprehensive review of the literature. For the lower truncated univariate normal, Cohen (1951a) proposes a recursive formula for its moments. In addition, Cohen (1951b) derives a recursive formula for the moments of the doubly truncated univariate normal. For \( n = 2 \), Rosenbaum (1961) provides the first two moments for the singly truncated case, and Khatri and Jaiswal (1963) provide a recurrence relation to obtain all the bivariate moments for the lower truncated case. For the doubly truncated case, Shah and Parikh (1964) and Dyer (1973) propose recurrence formulae for the bivariate moments. For the \( n \)-dimensional case, Birnbaum and Meyer (1953) derive a recursive formula for the bivariate moments in the lower truncated case. Gupta and Tracy (1976) provide a recurrence relation between different product moments of a doubly truncated univariate normal distribution.
multivariate normal. Unfortunately, since their recurrence relation does not express the product moments in terms of lower order product moments, it has been of little practical use besides the case of bivariate moments. Lee (1983) also presents a recurrence relation between product moments of a doubly truncated multivariate normal, but his relation requires the powers of all but one of the variables to be equal to one. Therefore, his formula cannot be used when some of the variables have powers greater than one. The moment generating function of the lower truncated multivariate normal distribution is available in Tallis (1961) and, in principle, it can be used to compute all the product moments for the lower truncated multivariate normal. Tallis (1961) provides explicit expressions of some low order moments for the $n = 2$ and $3$ cases. However, differentiating this moment generating function to obtain higher order moments involves tedious calculations. Recently, Arismendi (2013) overcomes this difficulty by providing explicit expressions for computing arbitrary order product moments. However, the required calculations for this approach can be quite time consuming. For example, when $n = 6$, computing all the fourth order moments, i.e., $k_1 + \cdots + k_6 = 4$, for the lower truncated multivariate normal distribution requires more than 5.4 hours on a PC with an Intel i7-4790K. In contrast, the Matlab program based on our algorithm computes all the product moments with $0 \leq k_i \leq 4$ ($i = 1, \ldots, 6$) in less than 29 seconds.

Instead of differentiating the moment generating function, we approach the problem by directly computing the moments of folded and truncated multivariate normal distributions, which require evaluating $n$-dimensional integrals that involve the multivariate normal density. We develop simple and efficient recurrence formulae for these multivariate integrals.\footnote{It is worth noting that the recurrence formulae we present rely on the ability to evaluate the cumulative distribution function of multivariate normal distributions. This can be done fairly accurately using numerical methods when $n$ is less than or equal to four, but Monte Carlo methods are required for higher dimensional cases.} In the most general case, the recurrence formula involves $3n + 1$ terms, but in many cases the number of terms can be reduced to $n + 1$. The rest of the article is organized as follows. Section 2 presents a recurrence formula for an integral that is essential for the evaluation of moments of folded and truncated multivariate normal distributions. Section 3 presents the results for the folded multivariate normal distribution. Section 4 presents the results for the truncated multivariate normal distribution. Explicit expressions for some low order moments of folded and truncated multivariate normal distributions are presented in the online appendix. Section 5 discusses possible extensions.

2 A Recurrence Relation for a Multivariate Integral

Suppose $X = (X_1, \ldots, X_n)^T \sim N(\mu, \Sigma)$, where $\mu = (\mu_1, \ldots, \mu_n)^T$ is the mean of $X$, $\Sigma = (\sigma_{ij})$ is the covariance matrix of $X$, and $\sigma_i^2 \equiv \sigma_{ii}$ stands for the variance of $X_i$. The
The joint density function of $X$ is
\[
\phi_n(x; \mu, \Sigma) = \frac{1}{(2\pi)^n |\Sigma|^{n/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}.
\]

The cumulative distribution function of $X$ is denoted as
\[
\Phi_n(x; \mu, \Sigma) = \int_{-\infty}^{x} \phi_n(y; \mu, \Sigma) dy,
\]
where we make use of the short-hand notation
\[
\int_a^b f(x) dx \equiv \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_n \cdots dx_1.
\]

When $\mu = 0$, we suppress the argument $\mu$ and simply write $\phi_n(x; \Sigma)$ and $\Phi_n(x; \Sigma)$. In addition, let
\[
L_n(a, b; \mu, \Sigma) = \int_a^b \phi_n(y; \mu, \Sigma) dy.
\]

Based on the inclusion-exclusion principle, this probability can be written as a linear combination of $2^n$ different values of $\Phi_n(\cdot; \mu, \Sigma)$, that is,
\[
L_n(a, b; \mu, \Sigma) = \sum_{i_1, \ldots, i_n \in \{0, 1\}} (-1)^{n-\sum_{j=1}^{i} i_j} \Phi_n((y_{i_1}, \ldots, y_{i_n})^T; \mu, \Sigma),
\]
where $y_{ij} = a_j$ if $i_j = 0$ and $y_{ij} = b_j$ if $i_j = 1$.

For the special case of univariate standard normal (i.e., $n = 1$, $\mu = 0$, $\sigma = 1$), we use $\phi(x)$ and $\Phi(x)$ to denote its density and cumulative distribution functions, respectively. In addition, for the standard bivariate normal (i.e., $n = 2$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$), we let $\phi_2(x_1, x_2; \rho)$ stand for $\phi_2(x; \Sigma)$ and $\Phi_2(x_1, x_2; \rho)$ stand for $\Phi_2(x; \Sigma)$, where $\rho$ is the correlation coefficient between $X_1$ and $X_2$.

For two $n$-vectors $x = (x_1, \ldots, x_n)^T$ and $\kappa = (k_1, \ldots, k_n)^T$, let $x^\kappa$ stand for $x_1^{k_1} \cdots x_n^{k_n}$. By $a(i)$ we denote a vector $a$ with its $i$th element removed. For a matrix $A$, we let $A_{i,j}$ stand for the $i$th row of $A$ with its $j$th element removed. Similarly, $A(i,j)$ stands for the matrix $A$ with its $i$th row and $j$th column removed. Finally, we let $e_i$ denote an $n$-vector with its $i$th element equal to one and zero otherwise.

The integral that we are interested in evaluating is
\[
F_n^\kappa(a, b; \mu, \Sigma) \equiv \int_a^b x^\kappa \phi_n(x; \mu, \Sigma) dx.
\]

The boundary condition is obviously $F_0^\kappa(a, b; \mu, \Sigma) = L_n(a, b; \mu, \Sigma)$. When $n = 1$, it is straightforward to use integration by parts to show that (with the arguments of $F_k^\kappa$ suppressed)
\[
\begin{align*}
F_0^1 &= \Phi(\beta) - \Phi(\alpha), \\
F_{k+1}^1 &= \mu F_k^1 + k\sigma^2 F_{k-1}^1 + \sigma \{ a^k \phi(\alpha) - b^k \phi(\beta) \} \quad (k \geq 0),
\end{align*}
\]
where $\alpha = (a-\mu)/\sigma$ and $\beta = (b-\mu)/\sigma$.\(^2\) When $n > 1$, we need a similar recurrence relation in order to compute $F_\kappa(a, b; \mu, \Sigma)$. The following theorem presents the required result. Lee (1983) also presents a similar recursive relation but his result can only be applied when $\kappa$ is in the form of $(1, \ldots, 1)^T + k_i e_i$, whereas our result allows for an arbitrary $\kappa > 0$.

**Theorem 1** For $n > 1$,

$$F^n_{\kappa+e_i}(a, b; \mu, \Sigma) = \mu_i F^n_{\kappa}(a, b; \mu, \Sigma) + e_i^T \Sigma c_\kappa \quad (i = 1, \ldots, n), \quad (1)$$

where $c_\kappa$ is an $n$-vector with $j$th element

$$c_{\kappa,j} = k_j F^n_{\kappa-e_j}(a, b; \mu, \Sigma) + a_j k_j^2 \phi_1(a_j; \mu_j, \sigma_j^2) F^{n-1}_{\kappa-j}(a, b; \mu_j, \sigma_j^2) - b_j k_j^2 \phi_1(b_j; \mu_j, \sigma_j^2) F^{n-1}_{\kappa-j}(a, b; \mu_j, \sigma_j^2),$$

and

$$\mu_j^a = \mu_j + \sigma_j^{-1} \left( a_j - \mu_j \right),$$

$$\mu_j^b = \mu_j + \sigma_j^{-1} \left( b_j - \mu_j \right),$$

$$\tilde{\Sigma}_j = \Sigma_{(j), (j)} - \frac{1}{\sigma_j^2} \Sigma_{(j), j} \Sigma_{j, (j)}.$$

When $k_j = 0$, the first term in (2) vanishes. When $a_j = -\infty$, the second term vanishes, and when $b_j = \infty$, the third term vanishes.\(^3\)

**Proof:** Taking the derivative of the multivariate normal density, we have

$$-\frac{\partial \phi_n(x; \mu, \Sigma)}{\partial x} = \Sigma^{-1} (x - \mu) \phi_n(x; \mu, \Sigma).$$

Multiplying each element on both sides by $x^\kappa$ and integrating $x$ from $a$ to $b$, we have (after suppressing the arguments of $F^n_{\kappa}$)

$$c_\kappa = \Sigma^{-1} \begin{bmatrix} F^n_{\kappa+e_1} - \mu_1 F^n_{\kappa} & F^n_{\kappa+e_2} - \mu_2 F^n_{\kappa} & \cdots & F^n_{\kappa+e_n} - \mu_n F^n_{\kappa} \\ \end{bmatrix}, \quad (3)$$

where the $j$th element of the left hand side is

$$c_{\kappa,j} = -\int_{a_{(j)}}^{b_{(j)}} x^\kappa \phi_n(x; \mu, \Sigma) \bigg|_{x_j = a_j}^{b_j} \, dx + \int_a^b k_j x^{\kappa-e_j} \phi_n(x; \mu, \Sigma) \, dx \quad (4)$$

\(^2\)Note that when $k = 0$, the term with the undefined component $F^1_{-1}$ vanishes and $F^1_1$ can be computed based only on $F^0_0$.

\(^3\)It is worth noting that Theorem 1 also holds for nonnegative real values $k_i$. However, the proposed recursion has greater practical value when the $k_i$’s are nonnegative integers since the boundary conditions can be evaluated in this case.

4
by using integration by parts. Using the fact that
\[
\phi_n(x; \mu, \Sigma)|_{x_j=a_j} = \phi_1(a_j; \mu_j, \sigma_j^2)\phi_{n-1}(x_{(j)}; \bar{\mu}_j^a, \bar{\Sigma}_j),
\]
\[
\phi_n(x; \mu, \Sigma)|_{x_j=b_j} = \phi_1(b_j; \mu_j, \sigma_j^2)\phi_{n-1}(x_{(j)}; \bar{\mu}_j^b, \bar{\Sigma}_j),
\]
we obtain
\[
\kappa_{\kappa,j} = k_j F_{\kappa-e_j}^n(a, b; \mu, \Sigma) + a_{k_j}^j \phi_1(a_j; \mu_j, \sigma_j^2) F_{\kappa_j}^{n-1}(a_{(j)}, b_{(j)}; \bar{\mu}_j^a, \bar{\Sigma}_j)
\]
\[- b_{b_j}^j \phi_1(b_j; \mu_j, \sigma_j^2) F_{\kappa_j}^{n-1}(a_{(j)}, b_{(j)}; \bar{\mu}_j^b, \bar{\Sigma}_j).
\]
When \(k_j = 0\), the last integral in (4) is equal to zero, and the first term of \(\kappa_{\kappa,j}\) drops out. When \(a_j \to -\infty\), \(a_{k_j}^j \phi_1(a_j; \mu_j, \sigma_j^2) \to 0\), so the second term of \(\kappa_{\kappa,j}\) drops out. Similarly, when \(b_j \to \infty\), the third term of \(\kappa_{\kappa,j}\) drops out. Finally, multiplying both sides of (3) by \(\Sigma\), we obtain (1). This completes the proof of Theorem 1.

It should be emphasized that Gupta and Tracy (1976) present a similar recurrence relation for the moments of a doubly truncated multivariate normal distribution. Besides the fact that they are dealing with the special case of \(a = 0\), the main difference is that their recurrence relation is essentially stated as
\[
c_{\kappa,j} = e^T_j \Sigma^{-1} \begin{bmatrix} F_{\kappa+e_1}^n - \mu_1 F_{\kappa}^n \\ F_{\kappa+e_2}^n - \mu_2 F_{\kappa}^n \\ \vdots \\ F_{\kappa+e_n}^n - \mu_n F_{\kappa}^n \end{bmatrix} (j = 1, \ldots, n).
\]
In this form, one cannot compute \(F_{\kappa}^n\) by using only lower order terms, and it is difficult to use this recursion in practice. Due to this unfortunate situation, no attempts have been made to use this recurrence relation to compute higher order moments of a truncated multivariate normal for \(n \geq 3\). We overcome this problem in Theorem 1 by multiplying both sides of (3) by \(\Sigma\). This delivers a simple way to compute \(F_{\kappa}^n(a, b; \mu, \Sigma)\) based on at most \(3n + 1\) lower order terms, with \(n + 1\) of them being \(n\)-dimensional integrals and the rest being \((n - 1)\)-dimensional integrals.

Although Theorem 1 is stated as a recurrence relation, it is better to avoid using a recursive function to compute \(F_{\kappa}^n(a, b; \mu, \Sigma)\). For speed gains, it is much more efficient to first compute all the necessary \((n - 1)\)-dimensional integrals \((F_{\nu}^{n-1}(a_{(j)}, b_{(j)}, \bar{\mu}_j^a, \bar{\Sigma}_j)) and \(F_{\nu}^{n-1}(a_{(j)}, b_{(j)}, \bar{\mu}_j^b, \bar{\Sigma}_j)\) for \(0 \leq \nu \leq \kappa(j), j = 1, \ldots, n)\) and then build up the entire table of \(F_{\nu}^n(a, b; \mu, \Sigma)\) for \(0 \leq \nu \leq \kappa\).

When all the \(a_i\)’s are \(-\infty\) or all the \(b_i\)’s are \(\infty\), the length of the recurrence relation is reduced to \(2n + 1\). When all the \(a_i\)’s are \(-\infty\) and all the \(b_i\)’s are \(\infty\), we have
\[
F_{\kappa}^n(-\infty, \infty; \mu, \Sigma) = E(X^\kappa),
\]
which is the product moments of \(X\). In this case, the recurrence relation is
\[
E(X^{\kappa+e_i}) = \mu_i E(X^\kappa) + \sum_{j=1}^{n} \sigma_{ij} k_j E(X^{\kappa-e_j}) \quad (i = 1, \ldots, n),
\]
and it is of length \( n + 1 \). This recurrence relation was obtained by Takemura and Takeuchi (1988) and Willink (2005).

Another case of special interest occurs when \( a_i = 0 \) and \( b_i = \infty \), \( i = 1, \ldots, n \). For this scenario, let

\[
I_n^\kappa(\mu, \Sigma) \equiv F_n^\kappa(0, \infty; \mu, \Sigma).
\]

The recurrence relation for \( I_n^\kappa \) can be written as

\[
I_{n+\kappa}(\mu, \Sigma) = \mu_i I_n^\kappa(\mu, \Sigma) + \sum_{j=1}^{n} \sigma_{ij} d_{\kappa, j} \quad (i = 1, \ldots, n),
\]

where

\[
d_{\kappa, j} = \begin{cases} 
    k_j I_{n-k}^\kappa(\mu, \Sigma) & (k_j > 0), \\
    \phi_1(\mu_j; \sigma_j^2) I_{n-1}^{\kappa-1}(\tilde{\mu}_j, \tilde{\Sigma}_j) & (k_j = 0),
\end{cases}
\]

with \( \tilde{\mu}_j = \mu_j - \Sigma_{(j), j} \mu_j / \sigma_j^2 \). The length of this recursion is only \( n + 1 \). For \( n = 1 \), our \( I_1^\kappa(\mu, \sigma_j^2) \) function is closely related to the \( I_k \) function of Fisher (1931), which is defined as

\[
I_k(\xi) = \frac{1}{(2\pi)^{k/2}} \int_0^\infty \frac{x^k}{k!} e^{-\frac{(x+\xi)^2}{2}} dx.
\]

It can be readily seen that \( I_k(\xi) = I_k(-\xi, 1)/k! \), and it satisfies the recurrence relation

\[
(k + 1)I_{k+1}(\xi) = -\xi I_k(\xi) + I_{k-1}(\xi) \quad (k \geq 1).
\]

### 3 Folded Multivariate Normal

The folded multivariate normal distribution is simply the distribution of \(|X|\), where \( X \sim N(\mu, \Sigma) \). In this section, we present the correct expression of the moment generating function of \(|X|\) as well as our approach for computing arbitrary order moments of \(|X|\).

Following Chakraborty and Chatterjee (2013), let

\[
S(n) = \{ s : s = (s_1, \ldots, s_n), \text{ with } s_i = \pm 1, \ i = 1, \ldots, n \}
\]

be a set of different combinations of \( n \) positive and negative signs. By defining \( \Lambda_s = \text{Diag}(s_1, \ldots, s_n) \), Chakraborty and Chatterjee (2013) show that the joint density of \( Y = |X| \) is

\[
f_Y(y) = \sum_{s \in S(n)} \phi_n(y; \mu_s, \Sigma_s) \quad (y \geq 0),
\]

where \( \mu_s = \Lambda_s \mu, \Sigma_s = \Lambda_s \Sigma \Lambda_s \), and the cumulative distribution function of \( Y \) is simply

\[
F_Y(y) = \Pr[-y \leq X \leq y] = L_n(-y, y; \mu, \Sigma) \quad (y \geq 0).
\]

Using the same derivations as in Tallis (1961), it is easy to show that

\[
\int_0^\infty e^{t^Ty} \phi_n(y; \mu_s, \Sigma_s) dy = \int_0^\infty \frac{1}{(2\pi)^{k/2}} |\Sigma_s|^{-1/2} e^{t^Ty - \frac{1}{2}(y - \mu_s)^T \Sigma_s^{-1} (y - \mu_s)} dy.
\]
It follows that the moment generating function of $Y$ is
\[ m_Y(t) = E(e^{t^T Y}) = \sum_{s \in S(n)} e^{t^T \mu_s + \frac{t^T \Sigma_s t}{2}} \Phi_n(\mu_s + \Sigma_s t; \Sigma_s). \]

While it is possible to differentiate $m_Y(t)$ to obtain the product moments of $Y$, it is much easier to compute the product moments of $Y$ using our $I^n_\kappa(\mu, \Sigma)$ function. Specifically, we have
\[ E(Y^\kappa) = \sum_{s \in S(n)} \int_0^\infty y^\kappa \phi_n(y; \mu_s, \Sigma_s) dy = \sum_{s \in S(n)} I^n_\kappa(\mu_s, \Sigma_s). \]

All we need is to evaluate $2^n$ different $I^n_\kappa(\mu_s, \Sigma_s)$ to obtain $E(Y^\kappa)$. Using our recurrence relation in (5), these calculations are very fast even for moderately large $n$. For example, when running our Matlab program on a PC with an Intel i7-4790K CPU, it takes 3.7 seconds to compute $E(Y^\nu)$ for $0 \leq \nu \leq (5, 5, 5, 5)^T$ when $n = 4$, and 45.2 seconds to compute $E(Y^\nu)$ for $0 \leq \nu \leq (5, 5, 5, 5)^T$ when $n = 5$.

The recurrence relation for $I^n_\kappa(\mu, \Sigma)$ can be used to obtain explicit expressions for the product moments of $Y$. In the online appendix, we provide explicit expressions for $E[Y^\kappa]$ up to the fourth order, which allow us to obtain the mean and covariance matrix of $Y$ as

\[ E[Y_i] = \mu_i \text{erf}\left(\frac{\tilde{\mu}_i}{\sqrt{2}}\right) + 2\sigma_i \phi(\tilde{\mu}_i), \]
\[ \text{Var}[Y_i] = \mu_i^2 + \sigma_i^2 - E[Y_i]^2, \]
\[ \text{Cov}[Y_i, Y_j] = (\mu_i \mu_j + \sigma_{ij})\{4\Phi_2(\tilde{\mu}_i, \tilde{\mu}_j; \rho_{ij}) - 2\Phi(\tilde{\mu}_i) - 2\Phi(\tilde{\mu}_j) + 1\} \]
\[ + 2\mu_i \sigma_j \phi(\tilde{\mu}_j) \text{erf}\left(\frac{\tilde{\mu}_j - \rho_{ij} \tilde{\mu}_i}{\sqrt{2}(1 - \rho_{ij}^2)^{1/2}}\right) + 2\mu_j \sigma_i \phi(\tilde{\mu}_i) \text{erf}\left(\frac{\tilde{\mu}_i - \rho_{ij} \tilde{\mu}_j}{\sqrt{2}(1 - \rho_{ij}^2)^{1/2}}\right) \]
\[ + 4\sigma_i \sigma_j (1 - \rho_{ij}^2) \phi_2(\tilde{\mu}_i, \tilde{\mu}_j; \rho_{ij}) - E[Y_i]E[Y_j], \]

where $\rho_{ij} = \sigma_{ij}/(\sigma_i \sigma_j)$, $\tilde{\mu}_i = \mu_i/\sigma_i$, and $\text{erf}(\tilde{\mu}_i/\sqrt{2}) = \Phi(\tilde{\mu}_i) + \Phi(-\tilde{\mu}_i)$ is the error function.

### 4 Truncated Multivariate Normal

The doubly truncated multivariate normal distribution is obtained by conditioning on $a \leq X \leq b$, where $X \sim N(\mu, \Sigma)$. Let $Z$ be the resulting truncated normal random vector with density function
\[ f_Z(z) = \frac{\phi_n(z; \mu, \Sigma)}{L_n(a, b; \mu, \Sigma)} \quad (a \leq z \leq b). \]

The cumulative distribution function of $Z$ is
\[ F_Z(z) = \frac{1}{L_n(a, b; \mu, \Sigma)} \int_a^z \phi_n(x; \mu, \Sigma) dx = \frac{L_n(a, z; \mu, \Sigma)}{L_n(a, b; \mu, \Sigma)} \quad (a \leq z \leq b). \]
Generalizing the results in Tallis (1961), it is easy to show that the moment generating function of \( Z \) is

\[
m_z(t) = E(e^{tZ}) = \frac{1}{L_n(a, b; \mu, \Sigma)} e^{t\mu \cdot t\Sigma t} L_n(a, b; \mu + \Sigma t, \Sigma).
\]

In principle, one could differentiate this moment generating function to obtain \( E(Z^\kappa) = E(X^\kappa \mid a \leq X \leq b) \), but for higher order moments, these calculations are extremely tedious, and the resulting expressions are not computationally efficient. Instead, we express \( E(Z^\kappa) \) in terms of our \( F^m_n(a, b; \mu, \Sigma) \) in Section 2 as

\[
E(Z^\kappa) = \frac{1}{L_n(a, b; \mu, \Sigma)} \int_a^b z^\kappa \phi_n(z; \mu, \Sigma) dz = \frac{F^m_n(a, b; \mu, \Sigma)}{L_n(a, b; \mu, \Sigma)}.
\]

Using our recurrence relation in Theorem 1, the computation of \( E(Z^\kappa) \) is very fast even for moderately large \( n \). For example, when running our Matlab program on a PC with an Intel i7-4790K CPU, it takes 0.97 second to compute \( E(Z^5) \) for \( 0 \leq \nu \leq (5, 5, 5)^T \) when \( n = 4 \), and 10.1 seconds to compute \( E(Z^5) \) for \( 0 \leq \nu \leq (5, 5, 5, 5)^T \) when \( n = 5 \).

Our algorithm allows for the possibility that \( a_i = -\infty, b_i = \infty \), or both \( a_i = -\infty \) and \( b_i = \infty \), i.e., no truncation on \( X_i \). When all the \( a_i \)'s are \( -\infty \) (\( b_i \)'s are \( \infty \)), we have the upper (lower) truncated multivariate normal distributions. For these special cases, we can express \( E(Z^\kappa) \) in terms of the \( I^m_n(\mu, \Sigma) \) function, which can be computed with a shorter recursion. We first provide an illustration of this method for the lower truncated multivariate normal distribution. In this scenario, we can write \( E(Z^\kappa) \) as

\[
E(Z^\kappa) = \frac{1}{\Phi_n(\mu - a; \Sigma)} \int_0^\infty (y + a)^\kappa \phi_n(y; \mu - a, \Sigma) dy = \frac{1}{\Phi_n(\mu - a; \Sigma)} \sum_{0 \leq \nu \leq \kappa} \frac{\kappa!}{\nu!} \nu^\kappa a^{\nu - \kappa} I^\nu_n(\mu - a, \Sigma),
\]

where \( \nu = (\nu_1, \ldots, \nu_n)^T \) and

\[
\binom{\kappa}{\nu} = \frac{k!}{\nu_1!(k - \nu_1)!}.
\]

This alternative expression shows that by using a binomial expansion, we can write \( E(Z^\kappa) \) as a linear combination of \( \prod_{i=1}^n(b_i + 1) \) different \( I^\nu_n(\mu - a, \Sigma) \). In computing \( I^\nu_n(\mu - a, \Sigma) \), all the \( I^\nu_n(\mu - a, \Sigma) \) with \( 0 \leq \nu \leq \kappa \) have already been computed. Therefore, no additional work is required besides summing up these terms.

Similarly, for the upper truncated case, we can write

\[
E(Z^\kappa) = \frac{1}{\Phi_n(b - \mu; \Sigma)} \sum_{0 \leq \nu \leq \kappa} \frac{\kappa!}{\nu!} \nu^\kappa b^{\nu - \kappa} (-1)^{\sum_{i=1}^n \nu_i} I^\nu_n(b - \mu, \Sigma).
\]
5 Conclusion

The results in this article can be easily generalized to the case of multivariate normal mixtures. Generalizing the results to multivariate elliptical distributions requires a lot more work. Although the product moments of multivariate elliptical distributions can be obtained from the product moments of multivariate normal distributions (see, for example, Berkane and Bentler (1986) and Maruyama and Seo (2003)), it is not clear how to obtain product moments of folded and truncated multivariate elliptical distributions. We leave this topic for future research.

SUPPLEMENTARY MATERIAL: The Matlab package ftnorm contains a set of programs to compute the moment expressions given in the article. The online appendix ftnormapp contains some supplementary results that are referred to in the article.

References


